4. Let α be a complex root of the irreducible polynomial $x^3 - 3x + 4$. Find the inverse of $\alpha^2 + \alpha + 1$ in $F(\alpha)$ explicitly, in the form $a + b\alpha + c\alpha^2$, $a, b, c \in \mathbb{Q}$.
5. Let $K = F(\alpha)$, where α is a root of the irreducible polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$. Determine the element α^{-1} explicitly in terms of α and of the coefficients a_i.
6. Let $\beta = \zeta\sqrt[3]{2}$, where $\zeta = e^{2\pi i/3}$, and let $K = \mathbb{Q}(\beta)$. Prove that -1 can not be written as a sum of squares in K.

3. The Degree of a Field Extension

1. Let F be a field, and let α be an element which generates a field extension of F of degree 5. Prove that α^2 generates the same extension.
2. Let $\zeta = e^{2\pi i/7}$, and let $\eta = e^{2\pi i/5}$. Prove that $\eta \notin \mathbb{Q}(\zeta)$.
3. Define $\zeta_n = e^{2\pi i/n}$. Find the irreducible polynomial over $\mathbb{Q}$ of **(a)** ζ_4, **(b)** ζ_6, **(c)** ζ_8, **(d)** ζ_9, **(e)** ζ_{10}, **(f)** ζ_{12}.
4. Let $\zeta_n = e^{2\pi i/n}$. Determine the irreducible polynomial over $\mathbb{Q}(\zeta_3)$ of **(a)** ζ_6, **(b)** ζ_9, **(c)** ζ_{12}.
5. Prove that an extension K of F of degree 1 is equal to F.
6. Let a be a positive rational number which is not a square in $\mathbb{Q}$. Prove that $\sqrt[4]{a}$ has degree 4 over $\mathbb{Q}$.
7. Decide whether or not i is in the field **(a)** $\mathbb{Q}(\sqrt{-2})$, **(b)** $\mathbb{Q}(\sqrt[4]{-2})$, **(c)** $\mathbb{Q}(\alpha)$, where $\alpha^3 + \alpha + 1 = 0$.
8. Let K be a field generated over F by two elements α, β of relatively prime degrees m, n respectively. Prove that $[K:F] = mn$.
9. Let α, β be complex numbers of degree 3 over $\mathbb{Q}$, and let $K = \mathbb{Q}(\alpha, \beta)$. Determine the possibilities for $[K:\mathbb{Q}]$.
10. Let α, β be complex numbers. Prove that if $\alpha + \beta$ and $\alpha\beta$ are algebraic numbers, then α and β are also algebraic.
11. Let α, β be complex roots of irreducible polynomials $f(x), g(x) \in \mathbb{Q}[x]$. Let $F = \mathbb{Q}[\alpha]$ and $K = \mathbb{Q}[\beta]$. Prove that $f(x)$ is irreducible in K if and only if $g(x)$ is irreducible in F.
12. **(a)** Let $F \subset F' \subset K$ be field extensions. Prove that if $[K:F] = [K:F']$, then $F = F'$. **(b)** Give an example showing that this need not be the case if F is not contained in F'.
13. Let $\alpha_1, \ldots, \alpha_k$ be elements of an extension field K of F, and assume that they are all algebraic over F. Prove that $F(\alpha_1, \ldots, \alpha_k) = F[\alpha_1, \ldots, \alpha_k]$.
14. Prove or disprove: Let α, β be elements which are algebraic over a field F, of degrees d, e respectively. The monomials $\alpha^i\beta^j$ with $i = 0, \ldots, d-1$, $j = 0, \ldots, e-1$ form a basis of $F(\alpha, \beta)$ over F.
15. Prove or disprove: Every algebraic extension is a finite extension.

4. Constructions with Ruler and Compass

1. Express $\cos 15°$ in terms of square roots.
2. Prove that the regular pentagon can be constructed by ruler and compass **(a)** by field theory, and **(b)** by finding an explicit construction.

3. Derive formula (4.12).
4. Determine whether or not the regular 9-gon is constructible by ruler and compass.
5. Is it possible to construct a square whose area is equal to that of a given triangle?
6. Let α be a real root of the polynomial $x^3 + 3x + 1$. Prove that α can not be constructed by ruler and compass.
7. Given that π is a transcendental number, prove the impossibility of squaring the circle by ruler and compass. (This means constructing a square whose area is the same as the area of a circle of unit radius.)
8. Prove the impossibility of "duplicating the cube," that is, of constructing the side length of a cube whose volume is 2.
9. **(a)** Referring to the proof of Proposition (4.8), prove that the discriminant D is negative if and only if the circles do not intersect.
   (b) Determine the line which appears at the end of the proof of Proposition (4.8) geometrically if $D \geq 0$ and also if $D < 0$.
10. Prove that if a prime integer p has the form $2^r + 1$, then it actually has the form $2^{2^k} + 1$.
11. Let C denote the field of constructible real numbers. Prove that C is the smallest subfield of $\mathbb{R}$ with the property that if $a \in C$ and $a > 0$, then $\sqrt{a} \in C$.
12. The points in the plane can be considered as complex numbers. Describe the set of constructible points explicitly as a subset of $\mathbb{C}$.
13. Characterize the constructible real numbers in the case that three points are given in the plane to start with.

*14. Let the rule for construction in three-dimensional space be as follows:
   (i) Three non-collinear points are given. They are considered to be constructed.
   (ii) One may construct a plane through three non-collinear constructed points.
   (iii) One may construct a sphere with center at a constructed point and passing through another constructed point.
   (iv) Points of intersection of constructed planes and spheres are considered to be constructed if they are isolated points, that is, if they are not part of an intersection curve.

   Prove that one can introduce coordinates, and characterize the coordinates of the constructible points.

5. Symbolic Adjunction of Roots

1. Let F be a field of characteristic zero, let f' denote the derivative of a polynomial $f \in F[x]$, and let g be an irreducible polynomial which is a common divisor of f and f'. Prove that g^2 divides f.
2. For which fields F and which primes p does $x^p - x$ have a multiple root?
3. Let F be a field of characteristic p.
   (a) Apply (5.7) to the polynomial $x^p + 1$.
   (b) Factor this polynomial into irreducible factors in $F[x]$.
4. Let $\alpha_1, \dots, \alpha_n$ be the roots of a polynomial $f \in F[x]$ of degree n in an extension field K. Find the best upper bound that you can for $[F(\alpha_1, \dots, \alpha_n) : F]$.